Introduction.

Let K be a number field and $\overline{K}$ a fixed algebraic closure of K. Let S be a finite set of places of K (including all the Archimedean places), and denote by $\mathcal{O}_S \subset K$ the ring

$$\{x \in K \mid v_{\mathfrak{p}}(x) \geq 0 \text{ for all finite primes } \mathfrak{p} \notin S\}$$

of S*-integers.*

Theorem. *The Diophantine equation $x + y = 1$ (the S-unit equation) admits only finitely many solutions for $x, y \in \mathcal{O}_S^\times$. Equivalently, the $\mathcal{O}_S$-points of the scheme*

$$\mathcal{Y} := \operatorname{Spec} \mathcal{O}_S[x, x^{-1}, (1-x)^{-1}] = \mathbf{P}^1_{\mathcal{O}_S} \setminus \{0, 1, \infty\}$$

is finite.

In 2020, Lawrence and Venkatesh gave an alternative proof of Faltings's thereom (i.e., the Mordell conjeture) using geometric arguments and tools from p-adic Hodge theory. The finiteness of the solutions to the S-unit equation serves as a good test case of their methods, in which all the machinery can be explicitly understood. The goal of this talk is to explain Lawrence-Venkatesh's proof of this theorem. This is also §4 of their paper.

Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a smooth projective map of $\mathcal{O}_S$-schemes serving as an integral model of a smooth projective map $X \to Y$ of K-schemes. Each point $y \in \mathcal{Y}(\mathcal{O}_S)$ extends to a point $y \in Y(K)$ via the natural embedding $\mathcal{Y}(\mathcal{O}_S) \to Y(K)$, and gives rise to a fiber X_y. The idea (which goes back to Parshin) that Faltings used in his proof of the Mordell conjecture is: to prove the finiteness of $\mathcal{Y}(\mathcal{O}_S)$, it suffices to prove the that there can only be finitely many such fibers.

We achieve this goal in two steps:

(A) each fiber X_y gives rise to a p-adic Galois representation, and there are only *finitely many* such representations;

(B) each such representation can only arise from *finitely many* fibers.

Step (A) is a theorem proved by Faltings. Step (B) is the meat of the Lawrence-Venkatesh method.

Finiteness of representation.

For the rest of the write-up, fix a rational prime p unramified in K and not below any primes in S. Let v be a prime finite place of K above p.

Consider the base change of X_y to $\operatorname{Spec}(\overline{K})$, denoted by $X_{y,\overline{K}}$. There is a natural Galois action G_K on $X_{y,\overline{K}}$ which gives rise to a p-adic Galois representation

$$\rho_y : G_K \to \mathrm{GL}(H^i_{\text{ét}}(X_{y,\overline{K}}, \mathbf{Q}_p))$$

on the p-adic étale cohomology of $X_{y,\overline{K}}$. To use p-adic Hodge theory, we need to restrict ρ_y to the local field G_{K_v} and consider the local version $\rho_{y,v}$ acting on the p-adic étale cohomology of $X_{y,\overline{K_v}}$. Our first source of finiteness is the finiteness of such Galois representations.

Proposition (Faltings)**.** *Fix $w, d \in \mathbf{Z}_{\geq 0}$. There are, up to conjugacy, only finitely many semisimple Galois representations $\rho : G_K \to \mathrm{GL}_d(\mathbf{Q}_p)$ satisfying:*

(i) ρ is unramified outside of S;

(ii) ρ is of pure weight w (i.e., for every prime $\mathfrak{p} \notin S$, all roots of the characteristic polynomial of the Frobenius $\mathrm{Frob}_{\mathfrak{p}}$ are algebraic numbers with complex absolute value $q_v^{w/2}$, where q_v is the cardinality of the residue field at $\mathfrak{p}$);

(iii) for every prime $\mathfrak{p} \notin S$, the characteristic polynomial of $\mathrm{Frob}_{\mathfrak{p}}$ has integer coefficients.

The proof is purely algebraic. We omit discussing it because it has nothing to do with the key idea of Lawrence-Venkatesh.

Somehow (I don't understand exactly why), the local p-adic Galois representation $\rho_{y,v}$ satisfy these conditions, so we've completed Step (A). In the rest of the notes we shall tackle Step (B).

The Gauss-Manin connection and period maps.

The Gauss–Manin connection.

Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a smooth proper morphism of $\mathcal{O}_S$-schemes. For each $q \geq 0$, define the *relative algebraic de Rham cohomology bundle*

$$\mathscr{H}^q := R^q \pi_* \Omega^\bullet_{\mathcal{X}/\mathcal{Y}}.$$

The fiber of this bundle over a point $y \in \mathcal{Y}(\mathcal{O}_S)$ is the q-th algebraic de Rham cohomology group

$$H^q_{\mathrm{dR}}(X_y/\kappa(y)) := \mathbf{H}^q(X_y, \Omega^\bullet_{X_y/\kappa(y)}).$$

Intuitively, a *connection* on a vector bundle tells us how to "differentiate" its sections along directions in the base. It prescribes a canonical way of comparing and identifying nearby fibers. A *Gauss–Manin connection* on $\mathscr{H}^q$ is a morphism of sheaves

$$\nabla : \mathscr{H}^q \longrightarrow \mathscr{H}^q \otimes \Omega^1_{\mathcal{Y}/\mathcal{O}_S},$$

satisfying the Leibniz rule. Katz and Oda constructed this connection as the boundary map arising from a short exact sequence of complexes

$$0 \to \Omega^{\bullet-1}_{\mathcal{X}/\mathcal{Y}} \otimes \pi^* \Omega^1_{\mathcal{Y}/\mathcal{O}_S} \to \Omega^\bullet_{\mathcal{X}/\mathcal{Y}} \to \Omega^\bullet_{\mathcal{Y}/\mathcal{O}_S},$$

where the middle term consists of all differentials on $\mathcal{X}$, the right term consists of "vertical" forms along the fibers, and the left term collects the "mixed" forms with one base differential component. The connecting morphism on higher direct images yields the desired ∇.

The analytic story.

We get a better intuition passing to the complex analytification of $\mathcal{X}$ and $\mathcal{Y}$ and work in the category of complex manifolds. In this setting, algebraic de Rham cohomology agrees with Betti cohomology: $H^*(X, \mathbf{C}) \cong H^*_{\mathrm{dR}}(X/\mathbf{C})$. Because $\Omega^\bullet_{\mathcal{X}/\mathcal{Y}}$ is a resolution of $\pi^{-1}\mathcal{O}_{\mathcal{Y}}$, we may write

$$R^q \pi_* \Omega^\bullet_{\mathcal{X}/\mathcal{Y}} \cong R^q \pi_*(\mathbf{C}_{\mathcal{X}}) \otimes_{\mathbf{C}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{Y}}.$$

Under this identification, the Gauss-Manin connection is defined by

$$\nabla(\alpha \otimes f) = \alpha \otimes df.$$

Heuristically, ∇ tells us how to differentiate cohomology classes as the base point y varies in $\mathcal{Y}$. Over a simply connected analytic chart $U \subset \mathcal{Y}$ with local coordinates $z_1, \ldots, z_m$, choose a basis of sections $s_1, \ldots, s_n$ of $\mathscr{H}^q$ on U. Then the connection can be written

$$\nabla s_i = \sum_j A_{ij}\, s_j, \qquad A_{ij} = \sum_k a_{ij,k}(z)\, dz_k,$$

where the $a_{ij,k}$ are holomorphic functions on U. A section $s = \sum_i f_i s_i$ is *flat* if $\nabla s = 0$, i.e.

$$A^T f = df.$$

In coordinates, this becomes the system of linear differential equations

$$\frac{\partial f}{\partial z_k} = A_k(z)\, f, \qquad k = 1, \ldots, m.$$

By the classical existence & uniqueness theorem for holomorphic ODEs, given an initial condition $f(0) = I$, there exists a unique holomorphic solution

$$f(z) = I + \sum_{|\alpha| \geq 1} c_\alpha z^\alpha,$$

which is a convergent power series near $z = 0$. Formally, the coefficients c_α can be computed recursively by substituting this expansion into the ODE and comparing coefficients. Hence the solution $f(z)$ is defined algebraically in the completed local ring $\mathbf{C}[[z_1, \ldots, z_m]]$, and the same series converges analytically on a small polydisk.

Therefore, the Gauss-Manin connection defines a canonical isomorphism

$$GM : H^q_{\mathrm{dR}}(X_{y_0}/\mathbf{C}) \xrightarrow{\simeq} H^q_{\mathrm{dR}}(X_y/\mathbf{C})$$

provided that y is sufficiently close to y_0 in the analytic topology. In coordinates, GM is given by the $n \times n$-matrix A with entries $A_{ij}(z_1, \cdots, z_m) \in \mathbf{C}[[z_1, \cdots, z_m]]$.

The p-adic story.

In the p-adic setting, we replace a contractible analytic neighborhood by a small v-adic ($v \mid p$) neighborhood of a point $y_0 \in \mathcal{Y}(\mathcal{O}_S)$, namely

$$\Omega_v = \{\, y \in \mathcal{Y}(\mathcal{O}_v) \mid y \equiv y_0 \mod v \,\},$$

where $\mathcal{O}_v$ is the ring of integers of the local field K_v. As in the complex case, the flatness condition yields a formal power series solution with coefficients in K_v. This is actually the same power series as in the complex case once an embedding $K_v \hookrightarrow \mathbf{C}$ is fixed, by the uniqueness of the solution. A direct computation shows that these series converge v-adically whenever $|z_i|_v < |p|_v^{1/(p-1)}$, where p is the residue characteristic of $\mathcal{O}_v$. Thus, the Gauss-Manin connection again produces a linear isomorphism

$$GM : H^q_{\mathrm{dR}}(X_{y_0}/K_v) \xrightarrow{\simeq} H^q_{\mathrm{dR}}(X_y/K_v)$$

whenever $y \in \mathcal{Y}(\mathcal{O}_v)$ satisfies $y \equiv y_0 \bmod v$.

The construction of the period map.

Fix $y_0 \in \mathcal{Y}(\mathcal{O}_S) \subset Y(K)$. Denote by X_0 the fiber of $X \to Y$ at y_0. We take for granted that $V = H^q_{\mathrm{dR}}(X_0/K)$ carries a Hodge filtration

$$V = F^0 V \supset F^1 V \supset \cdots \supset F^q V = 0$$

whose dimensions (the Hodge numbers) are independent of y_0. Let $\mathcal{H}$ be the flag variety parameterizing flags in V of length q with the same dimensional data as our Hodge filtration. Then, the Hodge filtration itself determines a point $h_0 \in \mathcal{H}(K)$.

Note that we can view y_0 as in $Y(\mathbf{C})$ and $Y(K_v)$ (after fixing embeddings of K into $\mathbf{C}$ and K_v). We will abuse the notation and still call it y_0 regardless of whether it is viewed as a K, K_v, or $\mathbf{C}$-point. Correspondingly, we have K_v-variety $\mathcal{H}_v$ and $\mathbf{C}$-varitey $\mathcal{H}_{\mathbf{C}}$.

Identifying $H^q_{\mathrm{dR}}(X_y/K_v)$ with $H^q_{\mathrm{dR}}(X_{y_0}/K_v)$ using the Gauss-Manin connection, we obtain a map

$$\Phi_v : \Omega_v \to \mathcal{H}_v(K_v)$$

sendind a point $y \in \Omega_v$ to the image of the Hodge filtration on $H^q_{\mathrm{dR}}(X_y/K_v)$ inside $H^q_{\mathrm{dR}}(X_{y_0}/K_v)$. This is called the ($p$-adic) *period map.*

Remark. Of course, if $\Omega_{\mathbf{C}}$ is a small complex-analytic neighborhood of y_0, then we have a period map $\Phi_{\mathbf{C}} : \Omega_{\mathbf{C}} \to \mathcal{H}_{\mathbf{C}}(\mathbf{C})$. The Hodge filtrations in the complex and p-adic cases are compatible, so if $y \in \Omega_v \cap \Omega_{\mathbf{C}}$, then since the GM isomorphisms are defined by the same power series, we conclude that the $\Phi_{\mathbf{C}}$ and Φ_v coincide. In (complex) Hodge theory, Griffiths transversality guarantees that the period map $\Phi_{\mathbf{C}}$ is a holomorphic map between complex manifolds.

Interlude: what p-adic Hodge theory tells us.

Observe that we were originally studying the p-adic étale cohomology of X_y as a Galois module, but the p-adic period map we constructed is based on the algebraic de Rham cohomology of X_y. The bridge between these two is given by p-adic Hodge theory. This is the first time I encounter p-adic Hodge theory been invoked in an essential way.

Let FL denote the category of triples $(W, \varphi, F^\bullet)$ called *filtered φ-modules* consisting of

- a K_v-vector space W,
- a Frobenius semilinear automorphism, meaning

  $$\varphi(aw) = \sigma(a)\varphi(w),$$

  for all $a \in K_v$ and $w \in W$, where $\sigma : K_v \to K_v$ is the Frobenius automorphism,
- and a descending filtration $F^\bullet$ on W

Then, p-adic Hodge theory gives a fully faithful functor from crystalline representations of G_{K_v} on $\mathbf{Q}_p$-vector spaces to FL. (A crystalline representation, roughly speaking, is a p-adic Galois representation that "comes from geometry" and behaves as well as possible with respect to reduction modulo p.) This is made precise in the following theorem.

Theorem (Crystalline comparison theorem)**.** *Let X be a smooth proper scheme over a p-adic field K with ring of integers $\mathcal{O}_K$, and let $\overline{K}$ be an algebraic closure. Assume X has good reduction at p. Then, for each integer $q \geq 0$, there is a canonical isomorphism*

$$D_{cris}(H^q_{\acute{e}t}(X_{\overline{K}}, \mathbf{Q}_p)) \xrightarrow{\simeq} (H^q_{\mathrm{dR}}(X/K), \varphi, F^\bullet)$$

of filtered φ-modules, where:

- *D_{cris} is Fontaine's functor from crystalline p-adic Galois representations to filtered φ-modules;*
- *$H^q_{\acute{e}t}(X_{\overline{K}}, \mathbf{Q}_p)$ is the p-adic étale cohomology with it natural G_K-action;*
- *$H^q_{\mathrm{dR}}(X/K)$ is the algebraic de Rham cohomology;*
- *φ is Frobenius action arising from the identification with crystalline cohomology over the special fiber (can you explain this part? I'm confused about this);*
- *$F^\bullet$ is the Hodge filtration on the de Rham cohomology.*

Note: p-adic Hodge theory only works with p-adic representations of local fields, not global fields. But that's good for us, because the tools we have to study the de Rham side, namely, the p-adic period maps Φ_v, are also local in nature (the domain of Φ_v is literally the v-adic neighborhood Ω_v). But our original problem was a global one. We fix this issue in the next section.

Reduce the problem from global to local.

Recall that our goal is to prove that each p-adic Galois representation can only arise from finitely many fibers X_y. This is a priori a global problem: we need to argue that the set of $y \in \mathcal{Y}(\mathcal{O}_S)$ satisfying $\rho_y \simeq \rho$ for a fixed ρ is finite. We shall show that this actually boils down to a local problem: after fixing a place $v \mid p$ of K, it suffices to show the finiteness of the set

$$A = A_{\rho, y_0} := \{y \in \mathcal{Y}(\mathcal{O}_S) \mid \rho_y \simeq \rho \text{ and } y \equiv y_0 \bmod v\}.$$

This is done through a series of elementary reductions. Let

$$U = \{t \in \mathcal{O}_S^\times \mid 1 - t \in \mathcal{O}_S^\times\}$$

be the solution set to the S-unit equation, whose finiteness we hope to prove. Let $U_1 \subset U$ denote the subset consisting of non-squares in $\mathcal{O}_S^\times$. It suffices to prove the finiteness of U_1: if $t = a^2$, then $1 - a^2 = (1-a)(1+a)$, so such cases reduce to smaller ones and can be handled separately.

We enlarge S and K so that S contains the primes above 2 and K contains the group of eighth roots of unity μ_8. This ensures that Kummer theory applies conveniently to all 2-power extensions of K. Let m be the largest power of 2 dividing the order of the group of roots of unity in K. By Kummer theory, for each $t \in K^\times$ the extension $K(t^{1/m})/K$ is cyclic of degree dividing m, and every cyclic m-extension of K arises in this way since K contains μ_m. Thus, to each $t \in U_1$ we can attach a cyclic extension $K(t^{1/m})/K$.

By Hermite–Minkowski, there are only finitely many number fields of a given degree that are unramified outside a fixed finite set S. Since all the fields $K(t^{1/m})$ are cyclic of fixed degree m and unramified outside S, only finitely many such L can occur. Hence, we may fix one such cyclic extension L/K and restrict our attention to the subset

$$U_{1,L} = \{\, t \in U_1 \mid K(t^{1/m}) \cong L \,\}.$$

Finally, we fix a good prime $v \notin S$. Reduction modulo v partitions $\mathcal{O}_S^*$ into finitely many residue classes in $(\mathcal{O}_S/v)$. We may therefore restrict further to a single congruence class

$$U_{1,L,t_0} = \{\, t \in U_{1,L} \mid t \equiv t_0 \pmod{v} \,\},$$

for some fixed $t_0 \in (\mathcal{O}_S/v)^\times$. Because there are only finitely many residue classes t_0, the finiteness of U will follow once we can show that each local subset U_{1,L,t_0} is finite. In the rest of the paper, we will always work with a fixed local field L, but we shall call it K_v.

Reduce to big monodromy using the period map.

We are reduced to showing the finiteness of the set

$$A = \{y \in \mathcal{Y}(\mathcal{O}_S) \mid \rho_y \simeq \rho \text{ and } y \equiv y_0 \bmod v\},$$

where ρ is a fixed p-adic representation ρ of G_{K_v} and y_0 is a fixed point of $\mathcal{Y}(\mathcal{O}_S/v)$. Using p-adic Hodge theory, we will instead study their corresponding φ-modules:

$$\rho : G_K \to \mathrm{GL}(H^i_{\text{ét}}(X_{y,\overline{K_v}}, \mathbf{Q}_p)) \rightsquigarrow (V_v, \varphi_v, \Phi_v(y)) \ .$$

Two representations ρ_y and $\rho_{y'}$ are isomorphic if

- their corresponding V_v and φ_v coincide,
- and there is some automorphism $T : V_v \to V_v$ interchanging the filtrations $\Phi_v(y)$ and $\Phi_v(y')$ and satisfy $T \circ \varphi_v = \varphi_v \circ T$.

Let $(V_v, \varphi_v, h_0^v), \cdots, (V_v, \varphi_v, h_N^v)$ be representatives of finitely many possible (due to the theorem of Faltings) isomorphism classes of ρ_y for $y \in A$. Then

$$\Phi_v(A) \subset \bigcup_{i=0}^{N} G \cdot h_i^v, \quad \text{where} \quad G = \{T \in \mathrm{Aut}_{K_v}(V_v) \mid T \circ \varphi_v = \varphi_v \circ T\}.$$

The condition that T commutes with φ_v is a little awkward to talk about algebraically, since φ_v is only semilinear and not linear. However, $\varphi_v^{[K_v:\mathbf{Q}_p]}$ is genuinely linear, and any T commuting with φ_v also commutes with every power of φ_v. Thus, we can consider the centralizer

$$Z = Z(\varphi_v^{[K_v:\mathbf{Q}_p]}) \supset G,$$

which is slightly bigger, but has the advantage of being an algebraic subgroup of $\mathrm{Aut}_{K_v}(V_v)$. Now, the finiteness of A follows from the finiteness of each

$$A_i := \Phi_v^{-1}(Z \cdot h_i^v).$$

This is in turn done by a dimensional argument: 0-dimensional analytic subsets of a rigid analytic space are finite! Observe that the Ω_v is a one-dimensional. (Intuitively: the analytification of $\mathcal{Y}$, a one-dimensional $\mathcal{O}_S$-scheme, over K_v is a one-dimensional rigid analytic manifold, and Ω_v can be thought of as an analytic neighborhood of some point on this one-dimensional rigid analytic manifold.) This means that we just need to show that A_i is properly contained in Ω_v. To do this, it suffices to prove

$$\dim Z < \dim \Phi_v(\Omega_v).$$

To achieve this, we will first find a lower bound on $\dim \Phi_v(\Omega_v)$. Perhaps surprisingly, this is where monodromy enters into the mix.

Consider the story in the complex setting first. Choosing a basepoint $y_0 \in \mathcal{Y}_{\mathbf{C}}(\mathbf{C})$, taking a fiber $V_{\mathbf{C}} = H^q_{\mathrm{dR}}(\mathcal{X}_{y_0}/\mathbf{C})$, and transporting cohomology classes along loops in $\mathcal{Y}_{\mathbf{C}}$ via flat sections of the Gauss-Manin connection gives a monodromy representation

$$\pi_1(\mathcal{Y}_{\mathbf{C}}, y_0) \to \mathrm{GL}(V_{\mathbf{C}}).$$

One also have the choice to invoke general theory: for a holomorphic vector bundle $\mathcal{E} \to M$, the Riemann-Hilbert correspondence gives an equivalence of categories between:

(i) local systems on M given by the sheaf of flat sections $\mathcal{L} = \ker \nabla$, where ∇ is a flat connection,

(ii) finite-dimensional representations of $\pi_1(\mathcal{M}, x_0)$.

Let $\Gamma \subset \mathrm{GL}(V_{\mathbf{C}})$ denote the Zariski closure of the image of this monodromy map.

Proposition. *Let $I_{\mathbf{C}}$ denote the Zariski closure of $\Phi_{\mathbf{C}}(\Omega_{\mathbf{C}}) \subset \mathcal{H}_{\mathbf{C}}$. Then $\Gamma \cdot h_0^\iota \subset I_{\mathbf{C}}$.*

Proof. The period map extends to a map $\widetilde{\mathcal{Y}}_{\mathbf{C}} \to \mathcal{H}_{\mathbf{C}}(\mathbf{C})$ from the universal cover, and this map is equivariant for the action of $\pi_1(\mathcal{Y}_{\mathbf{C}}, y_0)$. (Intuition: the period map cannot be extended to all of $\mathcal{Y}_{\mathbf{C}}$ because the GM connection only provides a canonical isomorphism between nearby fibers, but this obstruction disappears when lifted to the universal cover.) Now, since $\Phi_{\mathbf{C}}^{-1}(I_{\mathbf{C}})$ contains an open set, the image of $\Omega_{\mathbf{C}}$, in the universal cover, which is irreducible, it must be the entire space. Thus $\pi_1(Y_{\mathbf{C}}, y_0) \cdot h_0^\iota \subset I_{\mathbf{C}}$. The proposition follows by taking the Zariski closure. □

In the p-adic setting, since the same formal power series give both the complex and the p-adic period maps (see the remark at the end of the construction of the period map), the algebraic equations that cut out $\Phi_{\mathbf{C}}$ and Φ_v are identical. Therefore, we obtain an analogous lower bound on $\dim \Phi_v(\Omega_v)$, and we conclude that

$$\dim_{\mathbf{C}}(\Gamma \cdot h_0^\iota) \leq \dim_{K_v} \Phi_v(\Omega_v).$$

Now that we have a lower bound on $\dim \Phi_v(\Omega_v)$ given by the size of the monodromy, to show $\dim Z < \dim \Phi_v(\Omega_v)$, it suffices to show that $\dim Z < \dim_{\mathbf{C}}(\Gamma \cdot h_0^\iota)$. In other words, we need to find a smooth projective family $\mathcal{X} \to \mathcal{Y} = \mathbf{P}^1_{\mathcal{O}_S} \setminus \{0, 1, \infty\}$ with big monodromy.

A modified Legendre family.

The *Legendre family* over a K-variety Y is a family of elliptic curves

$$E_t : y^2 = x(x-1)(x-t),$$

parametrized by $t \in Y$. This gives a smooth proper morphism $\mathcal{E} \to Y$ whose fiber $\mathcal{E}_t$ is the elliptic curve E_t. For each fiber, the de Rham cohomology $H^1_{\mathrm{dR}}(E_t/K)$ is two-dimensional. (Note that we are only looking at H^1, because our Y is one-dimensional.) So

$$\dim Z \leq \dim(\mathrm{Aut}_K(H^1_{\mathrm{dR}})) = 4.$$

The period domain $\mathcal{H}$ is the flag variety classifying filtrations of two-dimensional vector spaces of type $(h^{1,0}, h^{0,1}) = (1,1)$, so the image of the period map has dimension 1. But $4 \not< 1$. Thus, we need to enlarge the fibers so that the period domain has higher dimension.

Consider the morphism $z \mapsto z^m : Y' \to Y$. We compose this map with a Legendre family over Y' to obtain a new smooth proper morphism $X \to Y$, whose fiber over $t \in Y$ is a disjoint union

$$X_t = \bigsqcup_{z^m = t} E_z,$$

where each E_z is an elliptic curve. Consequently,

$$H^1_{\mathrm{dR}}(X_t/K) \;\cong\; \bigoplus_{z^m=t} H^1_{\mathrm{dR}}(E_z/K),$$

which is $2m$-dimensional over K. The associated period domain $\mathcal{H}$ now parametrizes filtrations of a $2m$-dimensional space of type (m, m) (the Hodge filtration on the direct sum is simply the direct sum of the filtrations on each component), so it has dimension m.

What is $\dim Z$? This is a subtle point. A priori, H^1_{dR} is $2m$-dimensional. However, each fiber is defined over the extension $K(t^{1/m})$, and as a vector space over $K_v(t^{1/m})$, the dimension of H^1_{dR} is 2. Intuitively, although the total cohomology has $2m$ dimensions over K_v, the m components of the fiber become Galois-conjugate over $K_v(t^{1/m})$, merging into a single 2-dimensional space over the larger field. Therefore, we still have $\dim Z \leq 4$, as before.

(I realize here that my notation has not being the most consistent and clear. The reason that we can work with $K_v(t^{1/m})$ is given in the "reduce the problem from global to local" section.)

How big is the image of the monodromy map for our modified Legendre family? Let Γ be the Zariski closure of the image of ρ. Then

$$\Gamma \supset \prod_{z^m = t_0} \mathrm{SL}\big(H^1_{\mathrm{B}}(X_z, \mathbf{Q})\big).$$

In particular, the monodromy group acts with an open orbit on the period domain. This is because for the ordinary Legendre family, the image of the monodromy map contains the congruence subgroup $\Gamma(2)$, whose Zariski closure is SL_2 (this is computable, perhaps show computation). Now in our modified Legendre family, since each copy of $\mathcal{SL}_2$ acts transitively on $\mathbf{P}^1$, we have $\dim_{\mathbf{C}}(\Gamma \cdot h_0^\iota) = \dim(\mathbf{P}^1)^m = m$.