ON REALIZING RATIONAL AND POLYNOMIAL COHOMOLOGY RINGS

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ABSTRACT. This is an expository paper written during the 2021 REU program at the University of Chicago. This paper focuses on the following problem: which graded-commutative R-algebras can be realized as the cohomology ring of a space with coefficients in R? We provide an answer for the case $R = \mathbb{Q}$ using Quillen's model categories, after which we briefly outline the case for polynomial algebras over $\mathbb{Z}$ and $\mathbb{F}_p$ for odd prime p.

CONTENTS

1. INTRODUCTION

We focus on the following classification problem which was due to Hopf:

Question. Which graded-commutative R-algebra can be realized as the cohomology ring $H^*(X; R)$ of a space X with coefficients in R?

When $R = \mathbb{Q}$, the problem was solved in rational homotopy theory, which was developed by Daniel Quillen and Dennis Sullivan.

In 1967 Quillen introduced in [12] the concept of a model category, which is an axiomatic and homotopy-theoretical setting for homotopy theory. Later he applied this framework in his seminal paper [13] in which he constructed and proved the equivalence of homotopy theories of an algebraic category and a topological category. As a consequence, every differential graded Lie algebra over $\mathbb{Q}$ can be realized as the rational cohomology ring of a space. However, with Quillen's Lie algebra model it is difficult to do calculations.

Sullivan bridged the calculability gap in the 1970s using differential graded-commutative algebras as his algebraic model. He also generalized the idea of a de Rham complex and associated it with rational singular cohomology. In 1976 Bousfield and Gugenheim [5] redeveloped Sullivan's work using Quillen's model categories and the notion of Bousefield localization. We shall treat this in detail in the second section.

When $H^*(X;R)$ is a polynomial algebra over $\mathbb{Z}$ and $\mathbb{F}_p$, things get harder. In 1960 N. E. Steenrod attempted to answer this question. He proved in [15] using Steenrod operations that if $H^*(X;R) \cong \mathbb{Z}[x]$, then the degree of x must be 2 or 4. Further progress was made by J. F. Adams, C. W. Wilkerson, and W. G. Dwyer. The problem was completely solved by Kasper K. S. Andersen and Jesper Grodal in 2008 in [2] using p-compact groups. We sketch their proof in the third section.

2. The Rational Case

The main goal of this section is to show, restricting to rational and 1-connected objects of finite type, an equivalence between homotopy categories:

$$\mathrm{Ho}(\mathbf{Top}_{\mathbb{Q},1,f}) \cong \mathrm{Ho}(\mathbf{sSet}_{\mathbb{Q},1,f}) \cong \mathrm{Ho}(\mathbf{dgcAlg}_{\mathbb{Q},1,f}),$$

where **Top**, **sSet** and **dgcAlg** denote the category of topological spaces, simplicial sets, and differential graded-commutative algebras respectively. Therefore, any simply-connected differential graded-commutative algebra over $\mathbb{Q}$ of finite type can be realized as a rational cohomology ring $H^*(X;\mathbb{Q})$ of some space X.

To prove the desired equivalence, we need the concept of model categories, which we develop in Section 2.2. We will construct a pair of adjoint functors between **Top** and **sSet** and prove that they induce equivalence on the respective homotopy categories. In Section 2.3 we describe properties of differential graded-commutative algebras and the model structure on **dgcAlg**. Finally in Section 2.4 we construct and prove the similar equivalence between **sSet** and **dgcAlg**.

2.1. Rationalization and localization.

In this section we introduce some basic notions of rational homotopy theory. The main result is Corollary 2.10.

Definition 2.1. A 1-connected space is **rational** if $\pi_n(X)$ is a $\mathbb{Q}$-vector space for all $n \geq 1$. A map $f : X \to Y$ between simply-connected spaces is a **rational homotopy equivalence**, denoted $X \simeq_{\mathbb{Q}} Y$, if the map

$$\pi_n(f) \otimes \mathbb{Q} : \pi_n(X) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}$$

is an isomorphism for all $n \geq 1$.

Remark 2.2. In rational homotopy theory we study spaces up to rational homotopy equivalence. By tensoring with $\mathbb{Q}$ we "kill" the torsion abelian groups. In order for the tensor product to be well-defined, we need the homotopy groups to be abelian. That's why we will be working with simply-connected spaces exclusively. One can also choose to work with the more general notion of nilpotent spaces.

Definition 2.3. A collection $\mathfrak{C}$ of abelian groups is a **Serre class** if for a short exact sequence of abelian groups

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0\ ,$$

$A \in \mathfrak{C}$ and $C \in \mathfrak{C}$ implies $B \in \mathfrak{C}$. A map $f \in \mathrm{Hom}(A,B)$ is a **$\mathfrak{C}$-isomorphism** if both $\ker f \in \mathfrak{C}$ and $\mathrm{coker}\, f \in \mathfrak{C}$.

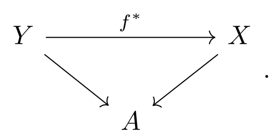
We can think of a Serre class as the class of groups we wish to "kill". In our case, we take $\mathfrak{C}$ to be the class of torsion abelian groups. Then f is a rational homotopy equivalence if and only if $\pi_n(f)$ is a $\mathfrak{C}$-isomorphism. The following two results are standard, modifying the classical Hurewicz theorem and Whitehead theorem.

Theorem 2.4 (Serre-Hurewicz)**.** *Let $\mathfrak{C}$ be a Serre class and X be a 1-connected space. Then $\pi_i(X) \in \mathfrak{C}$ for all $i < n$ if and only if $H_i(X) \in \mathfrak{C}$ for all $i < n$. Moreover, the Hurewicz map $\pi_n(X) \to H_n(X)$ is a $\mathfrak{C}$-isomorphism.*

Theorem 2.5 (Serre-Whitehead)**.** *Let $\mathfrak{C}$ be a Serre class and $f : X \to Y$ be a map between 1-connected spaces. Then $\pi_i(f) : \pi_i(X) \to \pi_i(Y)$ is a $\mathfrak{C}$-isomorphism for $i < n$ if and only if $H_i(f) : H_i(X) \to H_i(Y)$ is a $\mathfrak{C}$-isomorphism for $i < n$.*

We would like to associate a rational space $X_{\mathbb{Q}}$ to a given space X so that $X \simeq_{\mathbb{Q}} X_{\mathbb{Q}}$. We call this association **rationalization**, and we rationalize using localization. The concept of localization that we introduce below is similar to the notion of localization of rings in the context of algebra.

Definition 2.6. Let $\mathbf{C}$ be a category and L a subclass of morphisms in $\mathbf{C}$. An object A of $\mathbf{C}$ is **L-local** if for every $f \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$ such that $f \in \mathrm{L}$, the induced map $f^* : \mathrm{Hom}_{\mathbf{C}}(Y, A) \to \mathrm{Hom}_{\mathbf{C}}(X, A)$ is a bijection. Diagrammatically,



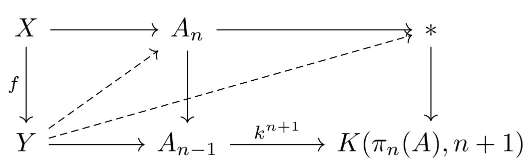
A map $l : A \to A'$ is an **L-localization** if $l \in \mathrm{L}$ and A' is L-local.

Now take L to be the class of rational homotopy equivalences. We show that L-localization is precisely the rationalization in the category of simply-connected spaces.

Proposition 2.7. *An object of* $\mathbf{Top}_1$ *is* L*-local if it is rational.*

Proof. Let A be a rational space and $f : X \to Y$ a rational homotopy equivalence between CW complexes. We show that $f^* : [Y, A] \to [X, A]$ is a bijection.

Surjectivity. Given any $X \to A$, we wish to find a lift $Y \to A$. Consider the Postnikov tower of A. By the 1-connectedness the base case $A_1 = *$, and the lift is trivial. Proceed by induction and suppose there is a lift $Y \to A_{n-1}$. Then the problem of finding $Y \to A_n$ amounts to finding a map $Y \to *$ below:



where k^{n+1} is the k-invariant. We only need to check that the obstruction class $\omega \in H^{n+1}(Y, X; \pi_n(A))$ is trivial. This is indeed the case since $\pi_n A$ is a $\mathbb{Q}$-vector space and $X \simeq_{\mathbb{Q}} Y$.

Injectivity. Given any $X \to A$ and suppose that $\lambda_0, \lambda_1 : Y \to A$, we wish to find a homotopy between λ_0 and λ_1. This is equivalent to another lifting problem:

$$\begin{array}{ccc} (X \times [0,1]) \cup_{X \times \partial[0,1]} (Y \times \partial[0,1]) & \longrightarrow & A \\ {\scriptstyle f'}\big\downarrow & \nearrow & \\ Y \times [0,1] & & \end{array}$$

where $X \times [0,1] \to A$ and $Y \times \partial[0,1] \to A$ are defined naturally. Note that f' is a rational homotopy equivalence, as it induces isomorphisms on all rational homology groups. Then the same obstruction argument in surjectivity applies. ☐

Proposition 2.8. *Every simply-connected space X admits a rationalization $X_{\mathbb{Q}}$.*

Proof. We will construct a rationalization explicitly and inductively using Postnikov tower on X. For an Eilenberg-MacLane space $K(G, n)$, the map

$$r : K(G, n) \to K(G \otimes \mathbb{Q}, n)$$

is a rationalization. Since X is simply-connected, we start the induction from $X_2 = K(\pi_2(X), 2)$ by taking $(X_2)_{\mathbb{Q}} = K(\pi_2(X) \otimes \mathbb{Q}, 2)$. Since $K(\pi_n(X) \otimes \mathbb{Q}, n+1)$ is rational, it is L-local by Proposition 2.7. Since X_n is the pullback of the path space over $K(\pi_n(X), n+1)$ along the k-invariant k^{n+1}, there is a bijection

$$[(X_{n-1})_{\mathbb{Q}}, K(\pi_n(X) \otimes \mathbb{Q}, n+1)] \cong [X_{n-1}, K(\pi_n(X) \otimes \mathbb{Q}, n+1)]$$

which identifies the k-th invariant k^{n+1} with a map $(k^{n+1})_{\mathbb{Q}}$. We thus define $(X_n)_{\mathbb{Q}}$ to be the pullback of the path space over $K(\pi_n(X) \otimes \mathbb{Q}, n+1)$ along $(k^{n+1})_{\mathbb{Q}}$, so that there is an induced map $r_n : X_n \to (X_n)_{\mathbb{Q}}$.

$$\begin{array}{ccccc} X_n & \longrightarrow & PK(\pi_n(X), n+1) & & \\ \big\downarrow & & \big\downarrow & & \\ X_{n-1} & \xrightarrow{k^{n+1}} & K(\pi_n(X), n+1) & & \\ & \searrow^{r_{n-1}} & & \searrow^{r} & \\ & (X_{n-1})_{\mathbb{Q}} & \xrightarrow{(k^{n+1})_{\mathbb{Q}}} & & K(\pi_n(X) \otimes \mathbb{Q}, n+1) \end{array}.$$

Finally, we define $X_{\mathbb{Q}}$ to be the inverse limit $\lim_n (X_n)_{\mathbb{Q}}$ of the Postnikov tower. One can verify that $X_{\mathbb{Q}}$ is indeed a rationalization of X. ☐

Now we prove the following converse of Proposition 2.7.

Proposition 2.9. *An object of* $\mathbf{Top}_1$ *is rational if it is* L*-local.*

Proof. Suppose A is L-local, then the rationalization $r : A \to A_{\mathbb{Q}}$ induces a bijection $[A_{\mathbb{Q}}, A] \cong [A, A]$. Since $A_{\mathbb{Q}}$ is rational, it is L-local by Proposition 2.7. Hence there is another bijection $[A_{\mathbb{Q}}, A_{\mathbb{Q}}] \cong [A, A_{\mathbb{Q}}]$. These two bijections together imply that $A \simeq_{\mathbb{Q}} A_{\mathbb{Q}}$. By the Serre-Whitehead theorem A is rational as well. ☐

Corollary 2.10. *Rationalization is equivalent to* L*-localization, where* L *is taken to be the class of rational homotopy equivalences.*

Proof. This follows from the definition, Proposition 2.7, and Proposition 2.9. ☐

2.2. **The first Quillen equivalence.** In this section we prove the equivalence of homotopy theories of topological spaces and simplicial sets. We begin by introducing simplicial sets.

Let $\mathbf{\Delta}$ be the category with order sets $[n] = \{0, 1, \cdots, n\}$ as objects and order-preserving functions $\varphi : [m] \to [n]$ (that is, functions satisfying $\varphi(i) \leq \varphi(i+1)$ for all $i \in [m]$) as morphisms.

Definition 2.11. A **simplicial object** in a category $\mathbf{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}$. A **cosimplicial object** is a functor $\mathbf{\Delta} \to \mathbf{C}$.

Simplicial and cosimplicial objects provides a combinatorial organization of things. For instance, Simplicial sets are simplicial objects in the category **Set** of sets. It generalizes the idea of simplicial complexes. A functor $\Phi : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Top}$ assigning each $[n]$ to a standard topological n-simplex Δ^n is an example of cosimplicial space.

Let **sSet** be the category with simplicial sets as objects and natural transformations between simplicial sets as morphisms. Then there is a natural functor $\Delta : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$ defined by $[n] \mapsto \Delta[n]$. We have the following diagram:

$$\begin{array}{ccc} \mathbf{\Delta}^{\mathrm{op}} & \xrightarrow{\quad\Phi\quad} & \mathbf{Top} \\ & \searrow^{\Delta} \quad \nearrow^{?} & \\ & \mathbf{sSet} & \end{array} .$$

How to construct a functor between **sSet** and **Top** based on Δ and Φ?

Definition 2.12. For any space X, the **singular simplicial complex** $\mathcal{S}_\bullet$ is a hom-functor $\mathbf{Top} \to \mathbf{sSet}$ given by

$$\mathcal{S}_\bullet(X) = \mathrm{Hom}_{\mathbf{Top}}(\Phi, X) = \mathrm{Hom}_{\mathbf{Top}}(\Delta^\bullet, X),$$

where $\Delta^\bullet$ denotes the standard topological simplices. Explicitly, the differentials of the complex is given by the following diagram:

$$\begin{array}{ccccc} [m] & \longrightarrow & \Delta^m & \longrightarrow & \mathcal{S}_m(X) = \mathrm{Hom}_{\mathbf{Top}}(\Delta^m, X) \\ \big\downarrow \varphi & & \big\downarrow \varphi_* & & \big\uparrow \varphi^* \\ [n] & \longrightarrow & \Delta^n & \longrightarrow & \mathcal{S}_n(X) = \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X) \end{array} .$$

For $\psi \in \mathcal{S}_n(X)$, the induced map φ^* is given by $\varphi^*(\psi) = \psi\varphi_*$. (The notation is cumbersome but every map is in the most natural sense possible.)

Definition 2.13. For any simplicial set S, define define its **geometric realization**

$$|S| = \coprod_{n \geq 0} (\Delta[n] \times \Delta^n)/\sim,$$

where $\Delta[n]$ are the n-simplices of S. An simplicial set is said to be **simply-connected** if its geometric realization is simply-connected.

In plain terms, we associate a topological n-simplex to each n-simplex of S and glue them together under the identification $(\varphi^*(x), t) \sim (x, \varphi_*(t))$.

Proposition 2.14. *Geometric realization is left adjoint to the singular simplicial set* $\mathcal{S}_\bullet$.

Proof. Let S be a simplicial set and X be a space. Then

$$\begin{aligned}\mathbf{sSet}(S, \mathcal{S}_\bullet(X)) &\cong \lim_{\Delta[n]\to S} \mathbf{sSet}(\Delta[n], \mathcal{S}_\bullet(X)) \\ &\cong \lim_{\Delta[n]\to S} \mathcal{S}_n(X) \\ &\cong \lim_{\Delta[n]\to S} \mathbf{Top}(\Phi[n], X) \\ &\cong \mathbf{Top}(|S|, X),\end{aligned}$$

where $\mathbf{sSet}(\Delta[n], \mathcal{S}_\bullet(X)) \cong \mathcal{S}_n(X)$ is given by the Yoneda lemma. □

We now turn to model categories. Model category abstracts certain common homotopy-theoretical notions that exist in many different categories.

Notation 2.15. For a class of morphisms M, we write M^l (M^r, respectively) for the class of morphisms that have the left (right, respectively) lifting property with respect to morphisms in M.

Definition 2.16. A **model category** if a category $\mathbf{C}$ with three subclasses of morphisms specified: the weak equivalences W, the fibrations Fib and the cofibrations Cof. They satisfy the following axioms:

(M1) The category $\mathbf{C}$ admits finite limits and colimits;
(M2) The class of weak equivalences W contains all isomorphisms and satisfy the 2-out-of-3 property: if any two of the three morphisms f, g, and fg belongs to W, then the third also belongs to W;
(M3) Any two classes determine the third as follows:

$$W = \mathrm{Fib}^r \circ \mathrm{Cof}^l, \quad \mathrm{Fib} = (\mathrm{W} \cap \mathrm{Cof})^r, \quad \mathrm{Cof} = (\mathrm{W} \cap \mathrm{Fib})^l;$$

(M4) Any morphism in $\mathbf{C}$ can be factored in two ways: either $\mathrm{Fib} \circ (\mathrm{W} \cap \mathrm{Cof})$ or $(\mathrm{W} \cap \mathrm{Fib}) \circ \mathrm{Cof}$.

An object A of $\mathbf{C}$ is is **fibrant** if $A \to \mathbf{1}$ is a fibration ad **cofibrant** if $\mathbf{0} \to A$ is a cofibration.

Example 2.17. Reader should be familiar with fibrations and cofibrations in **Top**. In fact, the following defines a model category structure on **Top**:

- weak homotopy equivalences as weak equivalences W;
- Serre fibrations as fibrations Fib;
- the set of maps $I = \{S^{n-1} \hookrightarrow D^n : n \geq 0\}$ generating cofibrations Cof.

This is sometimes referred to as the q-model structure on **Top**. The verification is not readily straightforward, and we refer to Quillen's original tome [12] for a detailed treatment.

There is a similar q-model structure on **sSet**, where weak equivalences are those morphisms $f : S \to T$ that induce weak homotopy equivalences $f_* : |S| \to |T|$ on the respective geometric realizations. The cofibrations are degreewise inclusions. To prove that these indeed define a modal category structure, see May-Ponto [9].

Definition 2.18. Let $\mathbf{C}$ and $\mathbf{D}$ be model categories. An adjunction $\mathcal{F} \dashv \mathcal{G}$ is a **Quillen adjunction** if it satisfy the following equivalent conditions:

(i) $\mathcal{F}$ preserves cofibrations and acyclic cofibrations;
(ii) $\mathcal{G}$ preserves fibrations and acyclic fibrations.

If $\mathcal{F} \dashv \mathcal{G}$ is a Quillen adjunction between $\mathbf{C}$ and $\mathbf{D}$, then the total derived functors $\mathbf{L}\mathcal{F}$ and $\mathbf{R}\mathcal{G}$ are defined by taking cofibrant and fibrant replacements:

$$\mathbf{L}\mathcal{F}(X) = \mathcal{F}(X^{\mathrm{cof}}), \quad \mathbf{R}\mathcal{G}(Y) = \mathcal{G}(Y^{\mathrm{fib}}).$$

These derived functors form an adjunction on the homotopy categories $\mathrm{Ho}(\mathbf{C})$ and $\mathrm{Ho}(\mathbf{D})$ defined by localizing with respect to weak equivalences:

$$\mathrm{Ho}(\mathbf{C}) = \mathbf{C}[\mathrm{W}_{\mathbf{C}}^{-1}], \quad \mathrm{Ho}(\mathbf{D}) = \mathbf{D}[\mathrm{W}_{\mathbf{D}}^{-1}].$$

Definition 2.19. An Quillen adjunction $\mathcal{F} \dashv \mathcal{G}$ is a **Quillen equivalence** if the following conditions hold:

(i) the derived adjunction unit $\bar{\eta} : X \to \mathbf{R}\mathcal{G}(\mathcal{L}(X))$ is a weak equivalence in $\mathbf{C}$;
(ii) the derived adjunction counit $\bar{\epsilon} : \mathbf{L}\mathcal{F}(\mathcal{G}(Y)) \to Y$ is a weak equivalence in $\mathbf{D}$.

Equivalently, $\mathcal{F} \dashv \mathcal{G}$ is a Quillen equivalence if the left derived functor $\mathbf{L}\mathcal{F} : \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$ is an equivalence of categories.

Proposition 2.20. *The singular simplicial complex $\mathcal{S}_\bullet$ and the geometric realization functor forms a Quillen equivalence between* **Top** *and* **sSet**.

Proof. This follows from the proof of the q-model structure on **sSet**. We omit the proof and refer to [9] for those interested. □

For a detailed treatment of model categories, see [9] and [7].

2.3. A model structure on $\mathbf{dgcAlg}_k$.

In this section we describe the algebraic model Sullivan used to model spaces. A key result will be Proposition 2.32.

In what follows, let R be a ring and k be a field with characteristic 0.

Definition 2.21. A **differential graded module** $M = \{M^n\}_{n \geq 1}$ over R is a R-module endowed with a cochain structure:

$$M^0 \xrightarrow{d} M^1 \xrightarrow{d} M^2 \xrightarrow{d} \cdots .$$

Write $\mathbf{DGM}_R$ for the category with differential graded modules over R as objects and chain maps as morphisms. We require chain maps to respect differentials.

Remark 2.22. Let M and N be differential graded modules over R. The (graded) tensor product $M \otimes_R N$ endows $\mathbf{DGM}_R$ with a symmetric monoidal structure. The symmetric braiding is given by

$$a \otimes b = (-1)^{kl} b \otimes a,$$

where k is the degree of a and l the degree of b. In addition, the Leibniz rule holds:

$$d(a \otimes b) = da \otimes b + (-1)^k a \otimes db.$$

Definition 2.23. A **differential graded-commutative algebra** A over R is a commutative monoid in $\mathbf{DGM}_R$.

Explicitly, A is an object of $\mathbf{DGM}_R$ together with a unit $\eta : R \to A$ and an associative product $\mu : A \otimes_R A \to R$ given by $a \otimes b \to a \cdot b$:

$$\begin{array}{ccccc}
(A \otimes_R A) \otimes_R A & \xrightarrow{\mu \otimes_R \mathrm{id}} & A \otimes_R A & & \\
\downarrow & & & \searrow^{\mu} & \\
 & & & & A \\
 & & & \nearrow_{\mu} & \\
A \otimes_R (A \otimes_R A) & \xrightarrow{\mathrm{id} \otimes_R \mu} & A \otimes_R A & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & M \otimes_R M & & \\
 & \nearrow^{\eta \otimes_R \mathrm{id}} & \downarrow{\scriptstyle \mu} & \nwarrow^{\mathrm{id} \otimes_R \eta} & \\
R \otimes_R M & \longrightarrow & M & \longleftarrow & M \otimes_R R
\end{array} .$$

The monoid is graded-commutative: $a \cdot b = (-1)^{kl} b \cdot a$. The object A is said to be **simply-connected** if $A^0 = k$ and $A^1 = 0$.

Let $\mathbf{dgcAlg}_R$ be the category with differential graded-commutative algebras as objects and chain maps as morphisms. Now we describe a model category structure on $\mathbf{dgcAlg}_k$.

Proposition 2.24. *The following defines a model category structure on* $\mathbf{dgcAlg}_k$*:*

(i) quasi-isomorphisms as weak equivalences W*;*
(ii) degreewise surjections as fibrations Fib*.*

Note that the cofibrations Cof *are thence uniquely determined to be* $(\mathrm{Fib} \cap \mathrm{W})^r$*.*

Before proving the theorem, we describe the models that mimic the notion of spheres S^n and disks D^{n-1} in **Top**. This is crucial for the later inductive arguments.

For a graded vector space V, write ΛV for the free graded-commutative algebra generated by V. It is equivalent to the tensor product

$$\mathrm{Sym}(V^{\mathrm{even}}) \otimes \mathrm{Etr}(V^{\mathrm{odd}})$$

of symmetric algebra on vectors of even degrees and exterior algebra on odd.

Construction 2.25. Let $S(n)$ be the free differential graded-commutative algebra with on one generator of degree n:

$$(\Lambda x, dx = 0), \quad |x| = n.$$

Let $D(n-1)$ be the free differential graded-commutative algebra $\Lambda(x, y)$:

$$(\Lambda(x, y), dx = 0, dy = x), \quad |x| = n, \quad |y| = n - 1.$$

Write $Z^n(A)$ for the space of n-cocycles of A. We make the following observations:

(i) $S(n) \hookrightarrow D(n-1)$ is a natural inclusion;
(ii) $\mathrm{Hom}_{\mathbf{dgcAlg}_k}(S(n), A) \cong Z^n(A)$;
(iii) $\mathrm{Hom}_{\mathbf{dgcAlg}_k}(D(n-1), A) \cong A^{n-1}$.

More generally, for any graded vector space V, define $S(V)$ to be ΛV with $dv = 0$ for all $v \in V$. Define $D(V)$ to be $\Lambda(V \oplus sV)$ with $dv = 0$ and $d(sv) = v$ for $v \in V$. Then similarly the following holds:

(i) $S(V) \hookrightarrow D(V)$ is a natural inclusion;
(ii) $\mathrm{Hom}_{\mathbf{dgcAlg}_k}(S(V), A) \cong \mathrm{Hom}_k(V, Z(A))$;
(iii) $\mathrm{Hom}_{\mathbf{dgcAlg}_k}(D(V), A) \cong \mathrm{Hom}_k(V, A)$.

We will see later that $S(n)$ and $D(n-1)$ functions like S^n and D^{n-1} in inductive gluing constructions.

Lemma 2.26. *For any graded vector spaces V and V' and any differential graded-commutative algebra A, consider the following morphisms:*

$$\begin{aligned} i : A &\to A \otimes_k S(V) \otimes_k D(V') \qquad & j : A &\to A \otimes_k D(V) \qquad l : S(V) \hookrightarrow D(V). \\ a &\mapsto a \otimes \eta \otimes \eta, & a &\mapsto a \otimes \eta, \end{aligned}$$

Then i and l are cofibrations, and j is a weak equivalence.

Proof of Proposition 2.24. The proof outlined here is taken from [7]. We check each of the axioms listed in Definition 2.16.

M1. We show that pushout and pullback exist in $\mathbf{dgcAlg}_k$. Let $f : B \to A$ and $g : C \to A$. The pullback of $B \to A \leftarrow C$ is $\{(b, c) : f(b) = g(c)\}$. The pushout of $B \leftarrow A \to C$ is $B \otimes_A C$, together with inclusions $\eta \otimes \mathrm{id} : C \to B \otimes_A C$ and $\mathrm{id} \otimes \eta : B \to B \otimes_A C$, where η denotes the unit.

M4. Let $f : A \to B$ be a morphism in $\mathbf{dgcAlg}_k$. We show that there are two ways to factor f:

$$f \in \mathrm{Fib} \circ (\mathrm{Cof} \cap \mathrm{W}), \quad \text{or} \quad f \in (\mathrm{Fib} \cap \mathrm{W}) \circ \mathrm{Cof}.$$

We prove $f \in (\mathrm{Fib} \cap \mathrm{W}) \circ \mathrm{Cof}$ by induction. Consider the base case

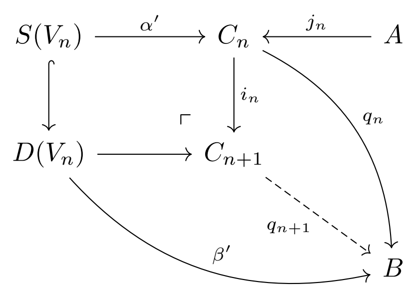
$$A \xrightarrow{j_0} C_0 = A \otimes_k S(Z(B)) \otimes_k D(B) \xrightarrow{q_0} B,$$

where $j_0 = \mathrm{id} \otimes \eta \otimes \eta$ and $q_o : a \otimes z \otimes b \mapsto f(a)\alpha(z)\beta(b)$ and α corresponds to $\mathrm{id} : Z(B) \to Z(B)$ by previous discussion. Then $j_0 \in \mathrm{Cof}$ by Lemma 2.26 and $q_0 \in \mathrm{Fib}$ by definition. But instead of $q_0 \in \mathrm{W}$, we can only guarantee that q_0 is surjective on cohomology.

To fix this issue, we add generators to C_n to kill cocycles in C_n. Suppose that $j_n : A \to C_n$ and $q_n : C_n \to B$ has already been constructed. Let

$$V_n = \{(c, b) \in C_n \oplus B : dc = 0 \text{ and } q_n(c) = db\} \text{ (with grading } |(c, b)| = |c|)$$

contain the information we want to erase. We add generators to C_n to construct C_{n+1} in analogy to adding generators to $S(V_n)$ to construct $D(V_n)$. Formally, we construct C_{n+1} as the pushout



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where α' is the composition $S(V_n) \to S(V_n') \to C_n$ and $V_n' = \mathrm{im}(V_n \to C_n)$. Let $j_{n+1} = i_n j_n : A \to C_{n+1}$. The map $q_{n+1} : C_{n+1} \to B$ comes from the universality of pushout. The map β' is the composition $D(V_n) \to D(V_n'') \to B$ where $V_n'' = \mathrm{im}(V_n \to B)$. This way, we have thus constructed $j_{n+1} : A \to C_{n+1}$ and $q_{n+1} : C_{n+1} \to B$.

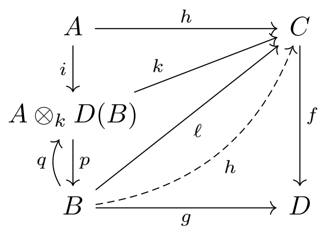
Finally let $C = \lim_n C_n$ and $j = \lim j_n : A \to C$. By Lemma 2.26 $\lambda \in \mathrm{Cof}$, and $i_n \in \mathrm{Cof}$. Hence $j_n \in \mathrm{Cof}$ for all n. Similarly let $q = \lim q_n : C \to B$, then q is surjective on cohomology. By our construction q is also injective on cohomology: any kernel in $H^\cdot(q_n)$ is becomes a boundary in C_{n+1}, and hence is trivial in C.

M3. We have forced $\mathrm{Cof} = (\mathrm{Fib} \cap \mathrm{W})^r$ in our construction. It remains to show that $\mathrm{Fib} = (\mathrm{Cof} \cap \mathrm{W})^l$ and $\mathrm{W} = \mathrm{Fib}^r \circ \mathrm{Cof}^l$. Let $f : A \to B$ and factorize f through $i : A \to C$ and $p : C \to B$ for $i \in \mathrm{Cof} \cap \mathrm{W}$ and $p \in \mathrm{Fib}$. If $f \in (\mathrm{Cof} \cap \mathrm{W})^r$, then there exists a lift $h : C \to A$ such that $p = fh$. Since p is surjective, f is surjective and $f \in \mathrm{Fib}$. Hence $(\mathrm{Cof} \cap \mathrm{W})^r \subseteq \mathrm{Fib}$.

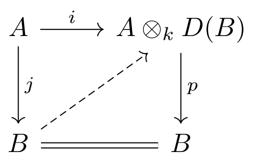
For the reverse inclusion, factorize $j : A \to B$ through

$$A \xrightarrow{i} A \otimes_k D(B) \xrightarrow{p} B.$$

If $j \in \mathrm{Cof} \cap \mathrm{W}$, then for any surjection $f : C \to D$ we construct a lift $h : B \to C$ as follows:



Since f is surjective, there exists a degreewise map $\ell : B \to C$ such that $g = f\ell$. Then $k : A \otimes_k D(B) \to C$ is determined by restricting it to h and ℓ on A and B respectively. The map $q : B \to A \otimes_k D(B)$ is constructed as the lift in the diagram



We define $h = kq$. It's standard to check that h is indeed the lift required. Hence $f \in (\mathrm{Cof} \cap \mathrm{W})^r$ and $\mathrm{Fib} \subseteq (\mathrm{Cof} \cap \mathrm{W})^r$.

The inclusion $\mathrm{W} \subseteq \mathrm{Fib}^r \circ \mathrm{Cof}^l$ is immediate. For the converse, we need to prove that $\mathrm{Cof}^r \subseteq \mathrm{W}$ and $\mathrm{Fib}^l \subseteq \mathrm{W}$. Factorize $f : A \to B$ through $i : A \to C$ and $p : C \to B$ where $i \in \mathrm{Cof}$ and $p \in \mathrm{Fib} \cap \mathrm{W}$. If $f \in \mathrm{Cof}^r$, then it follows that $f \in \mathrm{W}$. The other inclusion is proven similarly.

M2. All isomorphisms of cdg-algebras are quasi-isomorphisms. The 2-out-of-3 property holds for quasi-isomorphisms. □

We want to be able to describe the cofibrations in $\mathbf{dgcAlg}_k$ explicitly, instead of merely defining it to satisfy the axioms of model category.

Definition 2.27. A **relative Sullivan algebra** is an inclusion $A \hookrightarrow A \otimes_k \Lambda V$ of differential graded-commutative algebras where $V = \{V^i\}_{i \geq 1}$ is a differential graded module with filtration

$$0 = V(-1) \subseteq V(0) \subseteq V(1) \subseteq \cdots \bigcup_{n \geq 0} V(n) = V$$

satisfying $dV(n) \subseteq A \otimes_k \Lambda V(n-1)$ for all $n \geq 0$. A relative Sullivan algebra is **minimal** if $\operatorname{im} d \subseteq \Lambda^{\geq 2} V$, subspace of words of length at least two. A **Sullivan algebra** is a relative Sullivan algebra with $A = k$.

Lemma 2.28. *A* 1*-connected minimal Sullivan algebra can be filtered by degree.*

Proof. Take $V(n) = V^{\leq n}$. Since $\operatorname{im} d \subseteq \Lambda^{\geq 2} V$, for $v \in V^n$ suppose that $dv = x \cdot y$ for $x, y \in \Lambda V$. Then $|x| + |y| = |x \cdot y| = n + 1$. But $|x|, |y| \geq 2$ since ΛV is 1-connected. Hence $|x|, |y| \leq n - 1$, which means that $dV(n) \subseteq \Lambda V(n-1)$, and that ΛV is indeed a Sullivan algebra. □

A morphism $f : A \otimes_k \Lambda V \to A' \otimes_k \Lambda V'$ of relative Sullivan algebra is a morphism of differential graded-commutative algebras when restricted to the base algebras A:

$$\begin{array}{ccc} A & \xrightarrow{f|_A} & A' \\ \big\downarrow & & \big\downarrow \\ A \otimes_k \Lambda V & \xrightarrow{f} & A' \otimes_k \Lambda' \end{array} .$$

Remark 2.29. Relative Sullivan algebras can also be built inductively like CW complexes by gluing sphere algebras $S(V)$ along disk algebras $D(V)$. This is the notion of "adding generators to kill cocycles", as seen in the proof of Proposition 2.24. Explicitly, a relative Sullivan algebra $A \hookrightarrow X$ comes from a filtration

$$A = X(0) \subseteq X(1) \subseteq \cdots \bigcup_{n \geq 0} X(n) = X$$

where each $X(n)$ is built as the pushout of $D(V^n) \leftarrow S(V^n) \to X(n-1)$ for some differential graded module $V = \{V^n\}_{n \geq 1}$.

Proposition 2.30. *In the model category $\mathbf{dgcAlg}_k$, the relative Sullivan algebras are the cofibrations, and Sullivan algebras are cofibrant.*

Proof. We omit the proof and refer the readers to the Lemma 6.2.1 of [10]. □

Definition 2.31. Let $f \in \mathrm{Hom}_{\mathbf{dgcAlg}_k}(A, B)$. A **Sullivan model** for f is a relative Sullivan algebra $A \hookrightarrow M_B$ together with a quasi-isomorphism $M_B \simeq B$. Diagrammatically,

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & \searrow \quad \nearrow_{\simeq} & \\ & M_B & \end{array} .$$

Proposition 2.32. *Every $f \in \mathrm{Hom}_{\mathbf{dgcAlg}_{k,1}}(A, B)$ admits a minimal Sullivan model which is unique up to isomorphism.*

Proof. For the existence part, we build a minimal model ΛV for f inductively. For uniqueness, we show that it reduces to showing that any quasi-isomorphism of minimal algebra is an isomorphism.

Existence. Since B is 1-connected, let $V^0 = V^1 = 0$. Let $V^2 = H^2(B)$, then it defines a map $\Lambda V^{\leq 2} \to B$ in $\mathbf{dgcAlg}_{k,1}$. Suppose that $m_n : \Lambda V^{\leq n} \to B$ has already been constructed, then we define V^{n+1} by "adding generators to kill cocycles":

$$V^{n+1} = \left(\bigoplus k \cdot v_\alpha \right) \oplus \left(\bigoplus k \cdot v_\beta \right)$$

where $dv_\alpha = 0$ and $dv_\beta \in \ker H(m_n)$. Then we extend to $m_{n+1} : \Lambda V^{\leq n+1} \to B$ by letting $m_{n+1}(v_\alpha), d(m_{n+1}(v_\beta)) \in \operatorname{coker} H(m_n)$. This finishes the inductive process. The algebra ΛV obtained is naturally a minimal (the minimality is encoded in the fact that we are taking the filtration by degree) algebra. We only need to check that $\Lambda V \simeq B$. We omit the verification.

Uniqueness. Let M and M' be two minimal models for f. We first prove that $M \simeq M'$. This is a consequence of Proposition 2.30:

$$\begin{array}{ccc} A & \longrightarrow & M' \\ \big\downarrow & \nearrow & \big\downarrow \simeq \\ M & \xrightarrow[\simeq]{} & B \end{array} .$$

The dashed arrow $M \to M'$ exists since $A \hookrightarrow M$ is a cofibration. The lift is unique up to weak equivalence. Next, we wish to show that any quasi-isomorphism of minimal algebra is an isomorphism. For that we will need to develop the notion of homotopy theory between algebras (in terms of path objects). See [10]. □

2.4. **The second Quillen equivalence.** In this section we prove the equivalence

$$\mathrm{Ho}(\mathbf{sSet}_{\mathbb{Q},1,f}) \cong \mathrm{Ho}(\mathbf{dgcAlg}_{\mathbb{Q},1,f})$$

of homotopy theories. We begin by constructing a functor from **sSet** to $\mathbf{dgcAlg}_k$. For the whole section k will be a field with characteristic 0.

Definition 2.33. The **polynomial differential form** $\Omega^\bullet_{\text{poly}}$ is a simplicial object in $\mathbf{dgcAlg}_k$ (that is, a functor $\mathbf{\Delta}^{\text{op}} \to \mathbf{dgcAlg}_k$) given by

$$\Omega^\bullet[n] = \Omega^n_{\text{poly}} = \Lambda(t_0, \cdots, t_n, dt_0, \cdots, dt_n) / \left(\sum_{i=0}^n t_i = 1, \sum_{i=0}^n dt_i = 0 \right),$$

where $|t_i| = 0$ and $|dt_i| = 1$.

The superscript $^\bullet$ encodes the simplicial information specified by face maps $\delta_i : \Omega^n_{\text{poly}} \to \Omega^{n-1}_{\text{poly}}$ and degeneracy maps $\sigma_i : \Omega^n_{\text{poly}} \to \Omega^{n+1}_{\text{poly}}$ below:

$$\delta_i : t_k \mapsto \begin{cases} t_k, & k < i \\ 0, & k = i \\ t_{k-1}, & k > i \end{cases}, \qquad s_i : t_k \mapsto \begin{cases} t_k, & k < i \\ t_k + t_{k+1}, & k = i \\ t_{k+1}, & k > i \end{cases}.$$

In addition to the simplicial structure, $\Omega^\bullet_{\text{poly}}$ also carries a grading structure as an algebra. We encode this structure in the subscript and write $\Omega^\bullet_{\text{poly},p}$ for the k-module of homogeneous elements of $\Omega^\bullet_{\text{poly}}$ of degree p.

Lemma 2.34. *For any $p \geq 0$, $\Omega^\bullet_{\text{poly},p}$ is a weakly contractible Kan complex.*

Proof. This is a consequence of several theorems in simplicial homotopy theory. We state these theorems without proof.

Since $\Omega^\bullet_{\text{poly},p}$ is a simplicial k-module, it is a Kan complex. By a theorem of Eilenberg-Mac Lane, the homotopy groups of $\Omega^\bullet_{\text{poly},p}$ corresponds to the homology groups of the alternating face map complex

$$\cdots \longrightarrow \Omega^2_p \xrightarrow{\partial_2} \Omega^1_p \xrightarrow{\partial_1} \Omega^0_p$$

with differential given by $\partial_n = \sum_{i=1}^n (-1)^i \delta_i$. Define maps $s_n : \Omega^n_{\text{poly},p} \to \Omega^{n+1}_{\text{poly},p}$ by $s_n(1) = (1-t_0)^2$ and $s_n(t_i) = (1-t_0)t_{i+1}$ for $0 \leq i \leq n$. It's standard to verify that s_n is a degeneracy map. Finally, an extra degeneracy renders $\Omega^\bullet_{\text{poly},p}$ trivial. □

Now we can extend $\Omega^\bullet_{\text{poly}} : \mathbf{\Delta}^{\text{op}} \to \mathbf{dgcAlg}_k$ along $\mathbf{\Delta}^{\text{op}} \to \mathbf{sSet}$ via the left Kan extension. This establishes a functor $\mathbf{sSet} \to \mathbf{dgcAlg}_k$:

$$\begin{array}{ccc} \mathbf{\Delta}^{\text{op}} & \xrightarrow{\Omega^\bullet_{\text{poly}}} & \mathbf{dgcAlg}_k \\ & {\scriptstyle \Delta}\searrow \quad \nearrow & \\ & \mathbf{sSet} & \end{array}.$$

The process is remarkably similar to the one described in Section 2.2.

Definition 2.35. For any simplicial set S, the **PL de Rham complex** $\Omega^\bullet$ is a hom-functor $\mathbf{sSet} \to \mathbf{dgcAlg}_k^{\text{op}}$ given by

$$\Omega^\bullet(S) = \text{Hom}_{\mathbf{sSet}}(S, \Omega^\bullet_{\text{poly}}) = \lim_{\Delta[n]\to S} \Omega^n_{\text{poly}}.$$

The integration $\int$ maps smooth differential p-forms to p-cochains. By Stokes' theorem, $\int$ is a chain map between the de Rham complex and the singular cochain complex. The classical de Rham's theorem asserts that $\int$ induces isomorphism on cohomology. The analogy for polynomial forms is explain below.

Construction 2.36. Let S be a simplicial set and $x \in S$ a simplex. Let $f \in \Omega^p(S)$ be a polynomial p-form on S. Define an integration map $\oint_p : \Omega^p(S) \to C^p(S; k)$ by

$$(\oint_p f)(x) = \int_{\Delta[p]} f(x)dx = \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\cdots-x_p} f(x_1, \cdots, x_p)dx,$$

where $x = (x_1, \cdots, x_p)$ and $f(x) \in \mathbb{Q}[x_1, \cdots, x_p]$. The collection $\oint = \{\oint_p\}_{p\geq 0}$ defines a chain map from $\Omega^\bullet(S)$ to $C^\bullet(S; k)$ by the Stokes' theorem.

Proposition 2.37 (PL de Rham's theorem)**.** *For a simplicial set S, the integration map $\oint : \Omega^\bullet(S) \to C^\bullet(S; k)$ is a quasi-isomorphism.*

Proof. Since any simplicial set S has a skeletal filtration

$$\emptyset = S^{(-1)} \subseteq S^{(0)} \subseteq \cdots \subseteq S = \text{colim}_n S^{(n)}$$

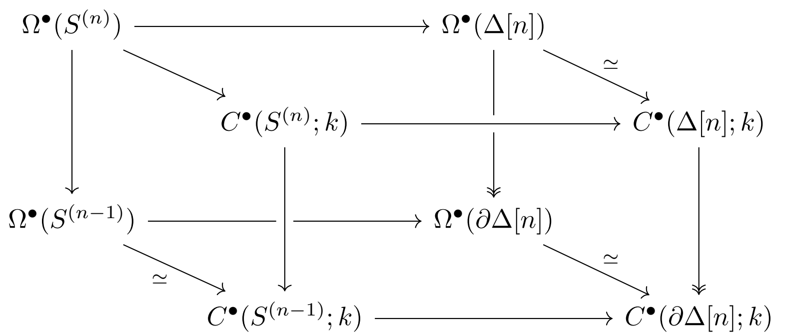
where $S^{(n)}$ is generated by non-degenerate simplices of S of degree at most n, we will perform induction on degrees for this proof.

First step. The goal is to prove the theorem for $S = \Delta[n]$ for any n. The limit construction gives $\Omega^\bullet(\Delta[n]) = \Omega^n_{\text{poly}}$, and the natural identification

$$\Lambda(t_0, \cdots, t_n, dt_0, \cdots, dt_n)/\left(\sum_{i=0}^n t_i = 1, \sum_{i=0}^n dt_i = 0\right) = \Lambda(t_1, \cdots, t_n, dt_1, \cdots, dt_n)$$

identifies Ω^n_{poly} with the tensor product of n copies of $\Lambda(t, dt) = \Omega^1_{\text{poly}}$. Since Ω^1_{poly} has trivial cohomology, so does Ω^n_{poly} by the Künneth formula. The chain map $\oint_n$ takes the trivial class in Ω^n_{poly} to the trivial class in $C^n(\Delta[n], k)$, thus inducing isomorphism on cohomology.

Second step. The base case $\Omega^\bullet(S^{(0)}) \simeq C^\bullet(S^{(0)}; k)$ is trivial. Suppose the claim is also true for $S^{(n-1)}$. Note that $S^{(n)}$ is obtained by gluing copies of $\Delta[n]$, which appears as a pushout of the form $\coprod \Delta[n] \leftarrow \coprod \partial\Delta[n] \to S$. We apply $\Omega^\bullet$ and $C^\bullet$ to this pushout square and obtain the following cube diagram:



The front and back squares are connected by $\oint$. They are both pullbacks squares since both $\Omega^\bullet$ and $C^\bullet$ are contravariant functors that preserve colimits. Moreover, since $\Delta[n] \to \partial\Delta[n]$ is a cofibration in **sSet**, the two vertical arrows on the right are fibrations in $\mathbf{dgcAlg}_k$.

We proved the quasi-isomorphism on the upper right corner in the first step. The quasi-isomorphisms on the bottom follow from the inductive assumption. By the cube lemma we obtain the fourth isomorphism $\Omega^\bullet(S^{(n)}) \simeq C^\bullet(S^{(n)}; k)$. □

We have thus constructed a functor $\Omega^\bullet : \mathbf{sSet} \to \mathbf{dgcAlg}_k^{\mathrm{op}}$. We now define its right adjoint functor. This is done similarly to Section 2.2.

Construction 2.38. For A a differential graded-commutative algebra over k, define $\mathcal{K}_\bullet$ to be the hom-functor $\mathbf{dgcAlg}_k^{\mathrm{op}} \to \mathbf{sSet}$ given by

$$\mathcal{K}_\bullet(A) = \mathrm{Hom}_{\mathbf{dgcAlg}_k}(A, \Omega^\bullet_{\mathrm{poly}}).$$

Then $\mathcal{K}_\bullet$ is right adjoint to $\Omega^\bullet$, as shown by the following calculation:

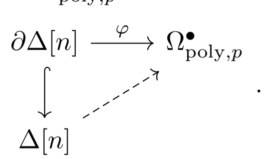
$$\begin{aligned}\mathbf{dgcAlg}_k(A, \Omega^\bullet(S)) &\cong \mathrm{colim}_{\Delta[n]\to S}\, \mathbf{dgcAlg}_k(A, \Omega^n_{\mathrm{poly}}) \\ &\cong \mathrm{colim}_{\Delta[n]\to S}\, \mathcal{K}_n(A) \\ &\cong \mathrm{colim}_{\Delta[n]\to S}\, \mathbf{sSet}(\mathcal{K}(A), \Delta[n]) \\ &\cong \mathbf{sSet}(\mathcal{K}(A), S),\end{aligned}$$

where $\mathcal{K}_n(A) \cong \mathbf{sSet}(\mathcal{K}(A), \Delta[n])$ is given by the co-Yoneda lemma.

Proposition 2.39. *The adjunction $\Omega^\bullet \dashv \mathcal{K}_\bullet$ is a Quillen adjunction.*

Proof. We show that $\Omega^\bullet$ preserves cofibrations and acyclic cofibrations. We only need to check this for generating cofibrations $\partial\Delta[n] \hookrightarrow \Delta[n]$ and generating acyclic cofibrations $\Lambda^i[n] \hookrightarrow \Delta[n]$.

Preserving cofibrations. We need to show that $\Omega^\bullet(\Delta[n]) \to \Omega^\bullet(\partial\Delta[n])$ is a degreewise surjections in $\mathbf{dgcAlg}_k$. For an element $\varphi \in \Omega^\bullet(\partial\Delta[n])$ of degree p, this amounts to finding a lift $\Delta[n] \to \Omega^\bullet_{\mathrm{poly},p}$:



By Lemma 2.34, $\Omega^\bullet_{\mathrm{poly},p}$ is a trivial Kan complex. Hence such lift always exists.

Preserving acyclic cofibrations. We need to show that $\Omega^\bullet(\Delta[n]) \to \Omega^\bullet(\Lambda^i[n])$ is a quasi-isomorphism. On the singular cochain level, $C^\bullet(\Delta[n]) \to C^\bullet(\Lambda^i[n])$ is a quasi-isomorphism. By the PL de Rham's theorem $\Omega^\bullet(\Delta[n]) \simeq C^\bullet(\Delta[n])$ and $\Omega^\bullet(\Lambda^i[n]) \simeq C^\bullet(\Lambda^i[n])$. Everything else follows. □

In PL de Rham's theorem we investigated the cohomology of $\Omega^\bullet$. In the following lemma we calculate the homotopy groups of $\mathcal{K}_\bullet$.

Lemma 2.40. *Let $A = \Lambda V$ be a 1-connected minimal algebra. Then there is a group isomorphism $\pi_n\mathcal{K}_\bullet(A) \cong \mathrm{Hom}_k(V^n, k)$. In particular if V is generated by one element of degree n, then $\mathcal{K}_\bullet(A)$ has the homotopy type of an Eilenberg-MacLane space $K(V^*, n)$, where V^* is the dual space of V.*

Recall that for a Quillen functor, its left and right derived functor are obtained by restricting to cofibrant and fibrant objects respectively. The following construction allows to compare the homotopy theories of **sSet** and $\mathbf{dgcAlg}_k$.

Construction 2.41. Since every object of **sSet** is cofibrant, the left derived functor

$$\mathbf{L}\Omega^\bullet : \mathrm{Ho}(\mathbf{sSet}) \to \mathrm{Ho}(\mathbf{dgcAlg}_k^{\mathrm{op}})$$

of $\Omega^\bullet$ corresponds with $\Omega^\bullet$. That is, $\mathbf{L}\Omega^\bullet(S) = \Omega^\bullet(S^{\mathrm{cof}}) = \Omega^\bullet(S)$.

Since every minimal model M_A is cofibrant in $\mathbf{dgcAlg}_k$ (and hence fibrant in the opposite category), the right derived functor

$$\mathbf{R}\mathcal{K}_\bullet : \mathrm{Ho}(\mathbf{dgcAlg}_k^{\mathrm{op}}) \to \mathrm{Ho}(\mathbf{sSet})$$

of $\mathcal{K}_\bullet$ is obtained by $\mathbf{R}\mathcal{K}_\bullet(A) = \mathcal{K}_\bullet(A^{\mathrm{fib}}) = \mathcal{K}_\bullet(M_A)$.

Finally, we wish to show that the adjunction $\Omega^\bullet \dashv \mathcal{K}_\bullet$ is a Quillen equivalence, so that the derived functors induces an equivalence on the homotopy categories. We show that by proving that the adjunction counit and unit are weak equivalences.

Definition 2.42. An object in **Top**, **sSet**, or $\mathbf{dgcAlg}_K$ is of **finite type** if the corresponding cohomology groups $H^n(X;\mathbb{Q})$, $H^n(|S|;\mathbb{Q})$, or $H^n(A)$ are finite-dimensional in each degree.

Remark 2.43. To restrict to the full subcategory of rational and 1-connected objects of finite type, and to use the minimal models described in Section 2.3, we first need to show that rational, 1-connected objects of finite type admit minimal models that are also rational, 1-connected, and finite. We leave the reader to check that this is indeed the case.

Proposition 2.44. *The derived adjunction counit $\bar{\epsilon} : \mathbf{L}\Omega^\bullet(\mathcal{K}_\bullet(A)) \to A$ is a weak equivalence in $\mathbf{dgcAlg}_{\mathbb{Q},1,f}$.*

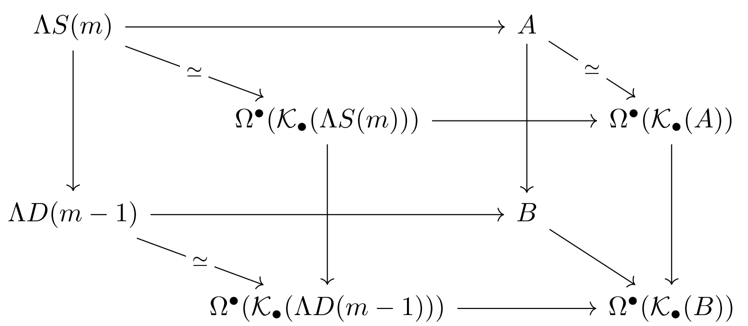
Proof. The derived adjunction counit is the composition of the normal adjunction counit ϵ with a cofibrant replacement:

$$\bar{\epsilon} : \mathbf{L}\Omega^\bullet(\mathcal{K}_\bullet(A)) = \Omega^\bullet(\mathcal{K}_\bullet(A)^{\mathrm{cof}}) \longrightarrow \Omega^\bullet(\mathcal{K}_\bullet(A)) \xrightarrow{\epsilon} A.$$

The cofibrant replacement is trivial since every object is cofibrant in **sSet**. The proof is by degreewise induction, similar to the proof of PL de Rham's theorem.

Base case. Let $A = \Lambda V$ be a minimal algebra with V generated by one element v of degree n. By Lemma 2.40 we have $\mathcal{K}_\bullet(A) \simeq K(\mathbb{Q}, n)$, and the cohomology of which is $\mathbb{Q}[x]$, the free graded-commutative algebra on one generator x. This can be calculated via spectral sequences, first replacing $K(\mathbb{Q}, n)$ with $K(\mathbb{Z}, n)$ since $\mathbb{Z} \hookrightarrow \mathbb{Q}$ induces rational homotopy equivalence. Let $z \in \Omega^\bullet(\mathcal{K}_\bullet(A))$ by a cycle representing x, and the map $\Omega^\bullet(\mathcal{K}_\bullet(A)) \to A$ defined by $z \mapsto v$ is a quasi-isomorphism.

Inductive step. Let A be a cofibrant object for which the theorem holds. If B is obtained by the pushout of $\Lambda D(m-1) \leftarrow \Lambda S(m) \to A$, we prove that the theorem also holds for B. Apply $\Omega^\bullet \mathcal{K}_\bullet$ to the pushout square to get a cube:



Since both $\Omega^\bullet$ and $\mathcal{K}_\bullet$ are contravariant, the front square is a homotopy pushout by the Eilenberg-Moore theorem. The back sqaure, which is a pushout, is a homotopy pushout since every object is cofibrant and $\Lambda S(m) \hookrightarrow \Lambda D(m-1)$ is a cofibration.

The bottom left quasi-isomorphism comes from the fact that both algebras are acyclic. The top left quasi-isomorphism follows from the base step, and the top right quasi-isomorphism is by the inductive assumption. By the cube lemma, the fourth map $B \simeq \Omega^\bullet(\mathcal{K}_\bullet(B))$.

Now we adapt the inductive step to our situation. First note that we can substitute A for its minimal model M_A, since every object in $\mathbf{dgcAlg}_{\mathbb{Q}}$ admits a unique minimal model. Moreover, M_A is 1-connected and of finite type if A has the same properties. We only need to prove that $\Lambda V \to \Omega^\bullet \mathcal{K}_\bullet(\Lambda V)$ is a quasi-isomorphism for a minimal model ΛV.

By Lemma 2.28 we can filter ΛV by degree, that is, $\Lambda V^n = \Lambda V(n)$. Suppose $V(n) = V(n-1) \oplus V'$ and that the theorem holds for $V(n-1)$, then for each generator $v \in V'$ of degree p, the inductive step claims that the theorem also holds for the pushout in the following diagram

$$\begin{array}{ccc} S(k) & \xrightarrow{t \mapsto dv} & \Lambda V(n-1) \\ \downarrow & & \downarrow \\ D(k-1) & \longrightarrow & \Lambda(V(n-1) \oplus \mathbb{Q} \cdot v) \end{array}.$$

Since ΛV is of finite type, repeating this procedure for finitely many times we see that the theorem also holds for $V(n)$. Finally, the map $H^i(\Lambda V) \to H^i(\Omega^\bullet \mathcal{K}_\bullet(\Lambda V))$ is same as the map $H^i(\Lambda V^{\leq i}) \to H^i(\Omega^\bullet \mathcal{K}_\bullet(\Lambda V^{\leq i}))$, which is an isomorphism. □

Proposition 2.45. *The derived adjunction unit* $\bar{\eta} : S \to \mathbf{R}\mathcal{K}_\bullet(\Omega^\bullet(S))$ *is a weak equivalence in* $\mathbf{sSet}_{\mathbb{Q},1,f}$.

Proof. The derived adjunction unit is the composition of the normal adjunction unit η with a fibrant replacement:

$$S \xrightarrow{\eta} \mathcal{K}_\bullet(\Omega^\bullet(S)) \longrightarrow \mathcal{K}_\bullet(\Omega^\bullet(S)^{\mathrm{fib}}) = \mathbf{R}\mathcal{K}_\bullet(\Omega^\bullet(S)) = \mathcal{K}_\bullet(M_{\Omega^\bullet(S)}).$$

In Proposition 2.39 we've shown that $\Omega^\bullet$ preserves weak equivalences. Hence $\bar{\eta}$ is a weak equivalence if and only if

$$\Omega^\bullet(\bar{\eta}) : \Omega^\bullet(S) \to \Omega^\bullet(\mathcal{K}_\bullet(M_{\Omega^\bullet(S)}))$$

is a quasi-isomorphism. Since $M_{\Omega^\bullet(S)}$ is a minimal algebra, by Proposition 2.44 $\Omega^\bullet(\mathcal{K}_\bullet(M_{\Omega^\bullet(S)})) \simeq M_{\Omega^\bullet(S)}$. Finally, $M_{\Omega^\bullet(S)} \simeq \Omega^\bullet(S)$ by definition of minimal model. Hence $\Omega^\bullet(\bar{\eta})$ is a quasi-isomorphism. □

Corollary 2.46. *The Quillen adjunction $\Omega^\bullet \dashv \mathcal{K}_\bullet$ is a Quillen equivalence.*

Proof. This follows from Proposition 2.44 and Proposition 2.45. □

Corollary 2.47. $\mathrm{Ho}(\mathbf{Top}_{\mathbb{Q},1,f}) \cong \mathrm{Ho}(\mathbf{sSet}_{\mathbb{Q},1,f}) \cong \mathrm{Ho}(\mathbf{dgcAlg}_{\mathbb{Q},1,f})$.

Proof. This follows from Proposition 2.20 and Corollary 2.46. □

3. The Polynomial Case

3.1. **Prime completion of finite loop spaces.** We introduce in this section the concept of p-compact groups.

Definition 3.1. Let G be a topological group and E a G-space. A **principal G-bundle** is a map $p : E \to B$ satisfying the following conditions:

(i) G acts trivially on B;
(ii) there is a cover $\mathcal{U}$ of B such that for any $U \in \mathcal{U}$ there is a G-homeomorphism $\varphi_U : p^{-1}(U) \to U \times G$ making the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi_U} & U \times G \\ {\scriptstyle p}\downarrow & \swarrow & \\ U & & \end{array} .$$

The second condition is referred to as the local triviality. Hence a principal G-bundle $E \to B$ consists of a locally trivial free G-space E with orbit space B.

Write $\mathcal{P}_G(X)$ for the isomorphism classes of principal G-bundles over X.

Lemma 3.2. *Let $\pi : E \to B$ be a principal G-bundle with E weakly contractible. Then there is an isomorphism $\varphi : [X, B] \to \mathcal{P}_G$ given by $f \mapsto f^*\pi$, where $f^*\pi$ is the pullback of π along f:*

$$\begin{array}{ccc} P & \longrightarrow & E \\ {\scriptstyle f^*\pi}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & B \end{array} .$$

In Lemma 3.2, π is a **universal** G-bundle and $B = \mathrm{B}G$ is a **classifying space** for G. Any topological group admits a classifying space, by Milnor's construction.

Remark 3.3. We need some mild conditions on the base space B. It's common to assume B as a CW complex. A weaker condition requires the locally trivialized bundle of B to have partition of unity. Equivalently, it's also fine to restrict B to paracompact spaces.

Definition 3.4. A **finite loop space** is a connected, pointed space $\mathrm{B}X$ such that $X \simeq \Omega \mathrm{B}X$ is homotopic equivalent to a finite CW complex.

Example 3.5. If X is a topological group G, then $\Omega \mathrm{B}G$ is indeed homotopy equivalent to G. This comes from the following comparison of fibre sequences:

$$\begin{array}{ccccc} G & \longrightarrow & EG & \longrightarrow & \mathrm{B}G \\ \downarrow & & \downarrow \simeq & & \| \\ \Omega \mathrm{B}G & \longrightarrow & P\mathrm{B}G & \longrightarrow & \mathrm{B}G \end{array},$$

where $EG \simeq P\mathrm{B}G$ because the path space $P\mathrm{B}G$ is weakly contractible.

For finite field $\mathbb{F}_p$, a map $f : X \to Y$ is an $\mathbb{F}_p$**-equivalence** if the induced map f^* is an isomorphism on $\mathbb{F}_p$-cohomology (mod-p cohomology):

$$f^* : H^\bullet(Y; \mathbb{F}_p) \xrightarrow{\cong} H^\bullet(X; \mathbb{F}_p).$$

Recall the notion of localization from Definition 2.6. In rational homotopy theory we localize with respect to rational homotopy equivalences. Here we define localization with respect to the class $\mathrm{F_p}$ of $\mathbb{F}_p$-equivalences.

With these in mind, we come to the object of main interest.

Definition 3.6. A p**-compact group** X is a finite loop space such that

(i) $\mathrm{B}X$ is $\mathrm{F_p}$-local;
(ii) $H^i(X, \mathbb{F}_p)$ is finite-dimensional for all i;

where $\mathrm{F_p}$ is the class of $\mathbb{F}_p$-equivalences.

The concept of a p-compact group was introduced by Dwyer and Wilkerson. It behaves similar to compact Lie groups, having concepts like maximal tori and Weyl groups. We will not go in detail of p-compact groups and their classifications.

3.2. **The arithmetic fracture theorem: from local to global.** In this section we see how to from $\mathbb{F}_p$-cohomology to integral cohomology and vice versa. All of the proofs are taken from [2].

First of all, by the *type* of a graded polynomial ring, we mean the information of its generators and their degrees. By *even type* we mean that the generators all have even degrees.

Proposition 3.7. *Let R be a commutative Noetherian ring of finite Krull dimension and $p \in R$ a non-unit prime. If $H^*(X; R)$ is a polynomial R-algebra of finite type, then $H^*(X; \mathbb{F}_p)$ is a polynomial $\mathbb{F}_p$-algebra of the same type.*

Proof. The key identity of the proof is

$$H^*(X; R) \otimes_R (R/pR) \cong H^*(X; R/p) \cong H^*(X; \mathbb{F}_p) \otimes_{\mathbb{F}_p} (R/pR).$$

Assuming the identity holds, then $H^*(X; \mathbb{F}_P) \otimes_{\mathbb{F}_p} (R/pR)$ is a graded polynomial (R/pR)-algebra (of the same type as $H^*(X; R)$). Since both $H^*(X; \mathbb{F}_p)$ and R/pR

are also $\mathbb{F}_p$-algebras, by an argument of commutative algebra, $H^*(X;\mathbb{F}_p)$ is a graded polynomial $\mathbb{F}_p$-algebra.

The first identity is ring-theoretical. Note that we already have the equivalence

$$C^*(X;R)\otimes_R (R/pR)\cong C^*(X;R/pR).$$

on the cochain level. Apply the universal coefficient theorem to both sides, then the Ext groups vanish since $H^*(X;R)$ is free over R. We thus only need to show that

$$H_*(X)\otimes_R (R/pR)\cong H_*(X\otimes_R (R/pR)).$$

Observe that $C^*(X;R)$ is a complex of flat modules. We now show that the submodules $d(C^*(X;R))$ are also flat. Since R has finite Krull dimension, by a result of Auslander and Buchsbaum, it also has finite finitistic flat dimension (the supremum of flat dimensions of all R-modules with finite flat dimension). Consider the short exact sequence

$$0\longrightarrow \ker d_n \longrightarrow C^n(X;R)\longrightarrow \operatorname{im} d_n \longrightarrow 0$$

where $\ker d_n$ and $\operatorname{im} d_{n+1}$ has the same flat dimension since $H_n(X)$ is flat. Since $C^n(X;R)$ is flat, either $\ker d_n$ and $\operatorname{im} d_n$ are also flat, or the flat dimension of $\ker d_n$ is larger than the flat dimension of $\operatorname{im} d_n$ by 1. But the second case is ruled out by the finiteness of finitistic flat dimension of R. Finally, since all submodules $d(C^*(X;R))$ are flat, the Künneth formula concludes the proof.

The second identity is obtained from the following line of identification:

$$\begin{aligned}H^*(X;R/p) &= H(\operatorname{Hom}_{\mathbb{F}_p}(C_*(X;\mathbb{F}_p),R/pR))\\ &\cong \operatorname{Hom}_{\mathbb{F}_p}(H_*(X;\mathbb{F}_p),R/pR)\cong H^*(X;\mathbb{F}_p)\otimes_{\mathbb{F}_p}(R/pR).\end{aligned}$$

□

Remark 3.8. The condition of R being a Noetherian ring with finite Krull dimension can be replaced by restricting X to be finite.

Write $X_p^\wedge$ for the p-completion of space X obtained by localizing with respect to $\mathbb{F}_p$-equivalences. The following proposition relates to p-compact groups.

Proposition 3.9. *If $H^*(X;\mathbb{F}_p)$ is a polynomial $\mathbb{F}_p$-algebra of finite type, then $X_p^\wedge \simeq \mathrm{B}Y$ is the classifying space of some p-compact group Y.*

In Proposition 3.7 we obtain information about the $\mathbb{F}_p$-cohomology of a space if its R-cohomology is known. The following proposition we attempt the converse: if the $\mathbb{F}_p$-cohomology is known, what other cohomologies do we know? For this much more difficult inverse problem, ring-theoretic argument as in Proposition 3.7 does not suffice, and we need homotopical argument.

Proposition 3.10. *Let I be a set of primes and J the set of primes not in I. If for each $p\in I$ there is a space X_p such that $H^*(X_p;\mathbb{F}_p)$ is a polynomial $\mathbb{F}_p$-algebra of finite even type, then there exists a 1-connected space Y of finite type such that $H^*(Y;\mathbb{Z}[J^{-1}])$ is a polynomial $\mathbb{Z}[J^{-1}]$-algebra of the same finite even type.*

Proof. Let B_p be $\mathbb{F}_p$-complete and $H^*(X_p;\mathbb{Z}_p)$ be a polynomial algebra over $\mathbb{Z}_p$ with generators of degrees $2d_1,2d_2,\cdots,2d_r$, where $\mathbb{Z}_p$ is the ring of p-adic integers. If $p>\max\{d_1,\cdots,d_r\}$, then by [1] X_p splits:

$$\pi_n(X_p)\cong \pi_{n-1}\left((S^{2d_1-1}\times\cdots\times S^{2d_r-1})_p^\wedge\right).$$

Let P be a set of primes and Y be the homotopy pullback of in the following:

$$\begin{array}{ccc} Y & \longrightarrow & \prod_{p\in P} X_p \\ \big\downarrow & & \big\downarrow \\ K & \xrightarrow{f} & (\prod_{p\in P} X_p)_{\mathbb{Q}} \end{array}$$

where X_p is $\mathbb{F}_p$-complete, $K = K(\mathbb{Z}[P^{-1}], 2d_1) \times \cdots \times K(\mathbb{Z}[P^{-1}], 2d_r)$, and f is induced by $\mathbb{Z}[P^{-1}] \to \mathbb{Q} \to (\prod_{p\in P} \mathbb{Z}_p) \otimes \mathbb{Q}$. We want to find a description of Y.

Apply Mayer-Vietoris sequence to the pullback square, we have the following long exact sequence:

$$\cdots \longrightarrow \pi_{n+1}((\textstyle\prod_{p\in P} B_p)_{\mathbb{Q}}) \longrightarrow \pi_n(Y) \xrightarrow{h} \pi_n(K) \times \pi_n(\prod_{p\in P} B_p) \longrightarrow \pi_n((\prod_{p\in P} B_p)_{\mathbb{Q}}) \longrightarrow \cdots .$$

We also know from a previous result that $\pi_n((\prod_{p\in P} B_p)_{\mathbb{Q}})$ is only nonzero for even n. If h is an isomorphism, then

$$\pi_n(Y) = \left(\bigoplus_{i, 2d_i = n} \mathbb{Z}\right) \oplus \left(\bigoplus_{p\in P} \mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, \pi_n(B_p))\right)$$

which is similar to the correct answer. The difference is $\mathbb{Z}$ instead of $\mathbb{Z}[P^{-1}]$ and the Tor group. □

Remark 3.11. A key step in the theorem is the arithmetic fracture theorem: there exists a homotopy pullback diagram:

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p \\ \big\downarrow & & \big\downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\prod_p X_p)_{\mathbb{Q}} \end{array} .$$

This is a remarkable theorem. Intuitively, one can rebuild X up to homotopy equivalence from its localizations, and hence the nickname "from local to global". See [9] for more.

Finally, the following result reduces the classification of polynomial algebras to the classification of p-compact groups, which was settled in [3].

Proposition 3.12. *If X is a p-compact group and $H^*(\mathrm{B}X; \mathbb{F}_p)$ is a polynomial $\mathbb{F}_p$-algebra of finite, even type, then*

$$H^*(\mathrm{B}X, \mathbb{F}_p) \cong H^*(\mathrm{B}G, \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*(\mathrm{B}Y; \mathbb{F}_p),$$

where G is a compact connected Lie group and Y is a product of exotic p-compact groups.

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